Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

Let $f,\ g \in \mathscr{S}\left(\mathbb{R}^d\right)$. Recall that in class we proved

$$\widehat{f * g} = (2\pi)^{\frac{d}{2}}\, \widehat{f}\widehat{g}. \tag{1}$$

Prove that

$$\widehat{f} * \widehat{g} = (2\pi)^{\frac{d}{2}}\, \widehat{fg}. \tag{2}$$

Hint: Consider the equivalent statement of (1) *for the inverse of the Fourier transform and apply it to* $\widehat{fg}$.

Exercise 2

Let $\mathcal{H}$ be an Hilbert space and V a closed linear subspace of $\mathcal{H}$.

a In class we proved that for any $f \in \mathcal{H}$ there exists an element $g_f \in V$ such that

$$\|f - g_f\| = \min_{h \in V} \|f - h\|. \tag{3}$$

Prove that g_f is the unique element of V that satisfies the minimum.

b In class we proved that g_f is such that $f - g_f \in V^{\perp}$. Prove that there is no other element $h \in V$ such that $f - h \in V^{\perp}$.

Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Prove that there exists a basis for $\mathcal{H}$. Prove moreover that $\mathcal{H}$ is separable if and only if there exists a countable base for it.

Hint: For the first part apply Zorn's Lemma to the set of (also infinite) orthonormal systems ordered by inclusion. Prove that any maximal orthonormal system is a base, i.e. is dense.

For the second part prove and use the following fact: if f *is an element of* $\mathcal{H}$ *and* S *is a basis for* $\mathcal{H}$*, there exists a sequence of elements* $\{e_n\}_{n\in\mathbb{N}} \subseteq S$ *such that* $f \in \overline{\operatorname{span}_{\mathbb{K}} \{e_n\}_{n\in\mathbb{N}}}$.

Exercise 4

Let $A,\ B$ bounded operators on an Hilbert space $\mathcal{H}$ and $\alpha,\ \beta \in \mathbb{C}$. Prove the following equalities:

$$\mathrm{id}^* = \mathrm{id} \tag{4}$$

$$(A^*)^* = A \tag{5}$$

$$(AB)^* = B^* A^* \tag{6}$$

$$(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*. \tag{7}$$

Moreover, prove that A^* is bounded and that $\|A^*\| = \|A\|$.